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An Optimal Reconstruction of Sampled Signals

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We indicate a pulse form supported in the sample interval, which optimizes the relation between low pulse energy and small error energy. © 1986 Academic Press, Inc

We consider the following situation. Given a sequence of binary rational numbers $\{f_k\}_{k\in\mathbb{Z}}$ arising as sample values of a function f in the Schwarz class $\mathscr{S}$ on the real axis with the support of $\hat{f}\frown(\hat{f}(\xi)=\int e^{-2\pi i x\xi}f(x)\,dx)$ in $[-\frac{1}{2},\frac{1}{2}]$. More precisely, we have the inequality $|f(k)-f_k|\leqslant 2^{-N}$, where 2^{-N} is the sampling accuracy.

The sampling theorem tells us that we may reconstruct f from the sample values. Let $\tau_k\delta$ denote the Dirac measure δ at $x=k$ and let $*$ denote convolution.

THE SAMPLING THEOREM. Assume that $f\in\mathscr{S}$ and $\operatorname{supp}\hat{f}\subset[-\frac{1}{2},\frac{1}{2}]$. Then holds

$$f=\frac{\sin\pi\cdot}{\pi\cdot}*\sum_k f(k)\,\tau_k\delta \qquad (\text{in } \mathscr{S}'). \tag{1}$$

We will give a proof of this well-known theorem, since we will have occasion to refer to the argument later.

Proof. Note that

$$\sum_k f(k)\,\tau_k\delta=f\sum_k\tau_k\delta \qquad (\text{in } \mathscr{S}'), \tag{2}$$

and that the Poisson Summation Formula may be written

$$\left(\sum_k\tau_k\delta\right)^{\wedge}=\sum_k\tau_k\delta \qquad (\text{in } \mathscr{S}'). \tag{3}$$

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Convolving with $\sin \pi \cdot / \pi \cdot$ in (2), taking Fourier transforms and using (3), we obtain

$$1_{[-1/2,1/2]} \cdot \left(\sum_k f(k)\, \tau_k \delta \right)^{\wedge} = \hat{f} \qquad (\text{in } \mathscr{S}'),$$

since $1_{[-1/2,1/2]} \hat{f} = \hat{f}$ and $(\sin \pi \cdot / \pi \cdot)^{\wedge} = 1_{[-1/2,1/2]}$. Inverse Fourier transformation then yields (1). ∎

Now, if we are given the sequence $\{f(k)\} \cap (\{f_k\}$ will be treated later) and we want to reconstruct the function f, then the Sampling Theorem tells us to create the Dirac δ measure. However, δ has infinite amplitude and energy. This makes it impossible for an engineer to build a circuit generating a δ. Of course, there are functions φ approximating δ. To fix our ideas, let us consider functions φ, which are even, nonnegative and which have $\int \varphi(x)\, dx = 1$ and $\operatorname{supp} \varphi \subset [-\frac{1}{2}, \frac{1}{2}]$. An examination of the argument in the proof of the Sampling Theorem shows that we may substitute $f \sum_k \tau_k \varphi$ for $\sum_k f(k)\, \tau_k \varphi$ in (1). However, in general, $f \sum_k \tau_k \varphi \neq \sum_k f(k)\, \tau_k \varphi$, the last sum being the one we get from the values $f(k)$. Thus the difference

$$\frac{\sin \pi \cdot}{\pi \cdot} * \left(f \sum_k \tau_k \varphi - \sum_k f(k)\, \tau_k \varphi \right)$$

represents the error in the reconstruction of f from the values $f(k)$.

Let us now estimate the error energy in an arbitrary sample interval $[k - \frac{1}{2}, k + \frac{1}{2}]$ before convolving with $\sin \pi \cdot / \pi \cdot$. (Convolution with $\sin \pi \cdot / \pi \cdot$ does not increase the energy norm, since this convolution corresponds to multiplication by $1_{[-1/2,1/2]}$ on the Fourier transform side and Parseval's Formula gives an isometry for the energy norms.) We estimate the error energy by

$$\int_{k-(1/2)}^{k+(1/2)} \{(f(x) - f(k))\, \varphi(x-k)\}^2\, dx \leqslant \sup_x |f'(x)|^2 \int x^2 \varphi^2(x)\, dx. \tag{4}$$

If we prescribe the maximal error energy, how small can we make the pulse energy $\int \varphi^2(x)\, dx$? The answer is given by an inequality of Uncertainty Relation type of Fritz Carlson's [1]; see also [2],

$$\left(\int \psi(x)\, dx \right)^4 \leqslant 4\pi^2 \int x^2 \psi^2(x)\, dx \int \psi^2(x)\, dx, \tag{5}$$

where ψ is an even, nonnegative function. (Translating (5) to Fourier transforms, we have, by Parseval's Formula,

$$\sup_{\xi} (\hat{\psi}(\xi))^4 \leqslant \int (\hat{\psi}'(\xi))^2 \, d\xi \int (\hat{\psi}(\xi))^2 \, d\xi, \tag{5'}$$

noting that, in this case, $\sup_\xi |\hat{\psi}(\xi)| = \hat{\psi}(0)$.) Equality in (5) holds for $\psi(x) = (1+x^2)^{-1}$ and its dilations $\psi_s(x) = \psi(x/s)$, $s > 0$.

For our pulse function φ, Carlson's inequality (5) implies an estimate for the pulse energy,

$$\int \varphi^2(x) \, dx \geqslant \frac{1}{4\pi^2 \varepsilon} \sup_x |f'(x)|^2, \tag{6}$$

where ε is the maximal error energy to the right in (4) (recall that $\int \varphi(x) \, dx = 1$). The pulse ψ_s, giving equality in (5), cannot be chosen as φ, since $\operatorname{supp} \psi_s = \mathbb{R}$. However, by choosing s small, we can take $\varphi = \text{const.} \, 1_{[-1/2, 1/2]} \cdot \psi_s$ and still nearly have minimal pulse energy in (6). Anyway, pulses are often built up digitally by approximation with step functions.

In conclusion then, choosing the pulse function φ as (s small)

$$\varphi(x) = \frac{1}{\pi} \cdot \frac{s}{s^2 + x^2} \cdot 1_{[-1/2, 1/2]}(x)$$

requires a nearly minimal pulse energy given a maximal error energy. Of course, smaller values of s have lesser energy but, on the other hand, $\sup_x \varphi(x) = \varphi(0) = 1/\pi s$.

A comparison to, e.g., the choice $\varphi = 1_{[-1/2, 1/2]}$ for the pulse function shows that this latter pulse (and its dilations $\varphi_\delta(t) = \varphi(t/\delta)$ with $\delta < 1$) requires more than three times the energy of the one just considered, given the same maximal error energy. This is seen by insertion into Carlson's inequality (5).

Finally, we indicate the possibility of taking the sampling error 2^{-N} into account. Then the maximal error energy in the arbitrary sampling interval $[k - \frac{1}{2}, k + \frac{1}{2}]$ is estimated by

$$\begin{aligned}
&\left(\int_{k-(1/2)}^{k+(1/2)} \{ (f(x) - f_k) \, \varphi(x-k) \}^2 \, dx \right)^{1/2} \\
&\quad \leqslant \left(\int_{k-(1/2)}^{k+(1/2)} \{ (f(x) - f(k)) \, \varphi(x-k) \}^2 \, dx \right)^{1/2} \\
&\qquad + \left(\int_{k+(1/2)}^{k+(1/2)} (f(k) - f_k) \, \varphi(x-k)^2 \, dx \right)^{1/2} \\
&\quad \leqslant \sup_x f'(x) \left(\int x^2 \varphi^2(x) \, dx \right)^{1/2} + 2^{-N} \left(\int \varphi^2(x) \, dx \right)^{1/2}.
\end{aligned}$$

Using Carlson's inequality (5), again a suitable pulse function may be chosen, e.g., minimizing the maximal error energy over any sampling interval.

References

1. F. Carlson, Une inégalité, *Ark. Mat., Astron. Fys. B* **25** (1) (1934), 1–5.
2. E. F. Beckenbach and R. Bellman, "Inequalities," Springer-Verlag, Berlin/Heidelberg/New York, 1965.